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A particle swarm optimization for solving joint pricing and lot-sizing problem with fluctuating demand and unit purchasing cost

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ABSTRACT

In this paper, we extend the classical economic order quantity model to allow for not only a function of price-dependent and time-varying demand but also fluctuating unit purchasing cost. The joint replenishment problem is subject to continuous decay and a general partial backlogging rate. The objective is to find the optimal replenishment number, time scheduling and periodic selling price to maximize the discounted total profit. An effective search procedure is provided to find the optimal solution by employing the properties derived in this paper and particle swarm optimization algorithm. Several numerical examples are used to illustrate the features of the proposed model.

1. Introduction

In many inventory systems, the deterioration of goods is a realistic phenomenon. It is well known that certain products such as medicine, volatile liquids, blood, foodstuffs and many others, decrease under deterioration (vaporization, damage, spoilage, dryness and so on) during their normal storage period. As a result, while determining the optimal inventory policy of that type of products, the loss due to deterioration cannot be ignored. The fundamental result in the development of economic order quantity model with deteriorating items is from Ghare and Schrader [1] who established the classical no-shortage inventory model with a constant rate of decay. Based on Ghare and Schrader's [1] model, researchers including Wu et al. [2], Huang and Yao [3], Huang and Liao [4], Mishra and Mishra [5], Maity and Maiti [6], Geetha and Uthayakumar [7] and Chang et al. [8] developed economic order quantity models that focused on deteriorating items.

Furthermore, in classical inventory models, the economic order quantity (EOQ) formula is assumed to be a constant demand rate and a fixed unit purchasing cost, and therefore offers ease of use and applications. In reality, the demand may vary with product life cycle duration. The assumption of a constant demand rate is usually valid in the maturity stage. In literature, the demand rate had been well approximated by specific forms to indicate the stage of a product in its life cycle. As Goyal and Giri [9] pointed out, most of the time-varying demand inventory models considered either linearly increasing/decreasing demand (i.e., $f(t) = a + bt$, with $a > 0, b \neq 0$) or exponentially increasing/decreasing demand (i.e., $f(t) = ae^{bt}$, with $a > 0, b \neq 0$) patterns. We refer the reader to their references therein for more details. Recently, Chen et al. [10], Chen et al. [11,12] dealt with the inventory model under the demand function following the product-life-cycle shape over a fixed time horizon.

Moreover, the assumption of a fixed unit purchasing cost does not reflect the situation where the inflation rate is high or the situation where price increase or decrease is expected. With the advances in technology and global division of labor, the

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unit cost of high-tech products might drop due to the introduction of new products. In the personal computer (PC) industry, Lee et al. [13] showed components' price declines constantly over the product life cycle. Under an exponential cost decrease but a constant demand rate, Khouja and Park [14] analyzed the problem of optimizing the lot size with equal length for the entire horizon. Teunter [15] then developed a net present value formulation of Khouja and Park's [14] model, and derived a simple modified EOQ formula. Khouja and Goyal [16] relaxed the restriction of equal length to allow varying cycle times.

Besides the continuous decrease of purchasing cost, in real-life situations, the gasoline price or raw material price may be going up constantly. When the cost of purchases as a percentage of sales is often substantial, it is necessary to include a fluctuating purchasing cost for the inventory system. Khouja et al. [17] developed the joint replenishment problem to analyse the effect of continuous unit purchasing cost decrease or increase on the optimal ordering frequencies. In contrast to the traditional EOQ model, Teng and Yang [18] assumed that both the demand function and the unit purchasing cost are fluctuating with time, which are more general than increasing, decreasing, and log-concave functions. Teng et al. [19] then provided an easy-to-use algorithm to find the optimal replenishment number and schedule for complete shortages. Teng and Yang [20] further allowed for time-varying purchasing cost and generalized holding cost over a finite-planning horizon.

From the competitive standpoint of the business, maximizing profit plays an important role for getting and keeping a successful position in a competitive market. However, the above inventory models subject to decreasing or increasing unit purchasing cost are developed to minimize the total relevant cost. To achieve profit maximization, Chen and Chen [21] presented an inventory model for a deteriorating item with a multivariate demand function of price and time but a fixed unit purchasing cost. Their model is solved by dynamic programming techniques to adjust the selling price upward or downward periodically. Chang et al. [22] established an inventory model for a retailer to determine its optimal selling price, replenishment number and replenishment schedule. They also assume a fixed unit purchasing cost, but the existence and uniqueness of the maximum solution is obtained under the same selling price per cycle. Similarly, when the unit purchasing cost is fluctuating with time, a decision maker needs to adjust its pricing strategy, but the joint pricing and replenishment policy is seldomly discussed.

In this paper, we assume that unit purchasing cost is positive and fluctuating with time. We investigate the replenishment policies for a deteriorating item with partial backlogging by considering a multivariate demand function of price and time and the effect of discount rate over multiperiod planning horizon. The fraction of unsatisfied demand backordered is any decreasing function of the waiting time up to the next replenishment. In addition, the selling price is allowed for periodical upward and downward adjustments. The objective of the inventory problem here is to determine the number of replenishments, the selling price per replenishment cycle, the timing of the reorder points and the shortage points. Following the properties derived from this paper, we provide a complete search procedure to find the optimal solutions by employing the search method based on particle swarm optimization algorithm. Several numerical examples are used to illustrate the features of the proposed model. At last, we make a summary and provide some suggestions for future research.

2. Assumptions and notation

The mathematical model in this paper is developed on the basis of the following assumptions and notations:

2.1. Assumptions

1. A single item is considered with a constant rate of deterioration over a known and finite planning horizon of length H.
2. The replenishment occurs instantaneously at an infinite rate.
3. There is no repair or replacement of deteriorated units during the planning horizon. The items will be withdrawn from the warehouse immediately as they deteriorate.
4. Shortages are allowed in all cycles and each cycle starts with shortages.
5. The fraction of shortages backordered is a decreasing function $\beta(x)$, where x is the waiting time up to the next replenishment, and $0 \le \beta(x) \le 1$ with $\beta(0) = 1$. Note that if $\beta(x) = 1$ (or 0) for all x, then shortages are completely backlogged (or lost).

2.2. Notation

$n =$ The number of replenishment cycles during the planning horizon (a decision variable)
$\theta =$ the deterioration rate
$r =$ the discount rate
$c_f =$ the ordering cost per order
$c_v(t) =$ the unit purchasing cost at time t, where $c_v(t)$ is a positive and continuous function of time in the planning horizon
$p_i =$ the selling price per unit (a decision variable) in the ith replenishment cycle, defined in the interval $[0, p_u]$
$f(t, p_i) =$ the demand rate at time t and price p_i with $f(t, p_i) = g(t)A(p_i)$, where $g(t)$ is a positive and continuous function of time in the planning horizon and $A(p_i)$ is any non-negative, continuous, convex, decreasing function of the selling price in $[0, p_u]$

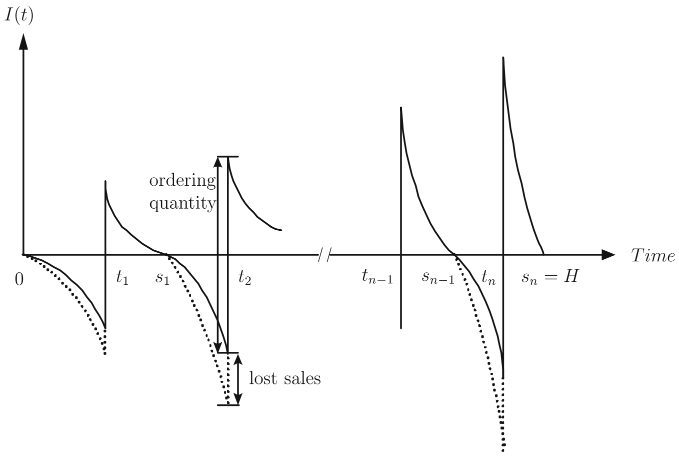


Fig. 1. Graphical representation of inventory system.

$c_h =$ the inventory holding cost per unit per unit time
$c_s =$ the backlogging cost per unit per unit time due to shortages
$c_l =$ the unit cost of lost sales
$t_i =$ the ith replenishment time (a decision variable), $i = 1, 2, \ldots, n$
$s_i =$ the time at which the inventory level reaches zero in the ith replenishment cycle (a decision variable), $i = 1, 2, \ldots, n-1$.

As a result, the decision problem here has $3n$ decision variables.

3. Model formulation

According to the notations and assumptions mentioned above, the behavior of the inventory system at any time can be depicted in Fig. 1. From Fig. 1, it can be seen that the depletion of the inventory occurs due to the combined effects of the demand and the deterioration during the interval $[t_i, s_i)$ of the ith replenishment cycle. Hence, the variation of inventory with respect to time can be described by the following differential equation:

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -f(t, p_i) - \theta I(t), \quad t_i < t < s_i, \tag{1}$$

with boundary condition $I(s_i) = 0$, $i = 1, 2, \ldots, n$. Solving the differential equation (1), we have

$$I(t) = \mathrm{e}^{-\theta t} \int_t^{s_i} \mathrm{e}^{\theta u} f(u, p_i)\,\mathrm{d}u, \quad t_i \le t < s_i. \tag{2}$$

On the other hand, the depletion of inventory occurs due to the demand backlogged during the interval $[s_{i-1}, t_i)$. The variation of inventory with respect to t can be described by the following differential equation:

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -\beta(t_i - t) f(t, p_i), \quad s_{i-1} < t < t_i, \tag{3}$$

with boundary condition $I(s_{i-1}) = 0$, $i = 1, 2, \ldots, n$. Solving the differential equation (3), we have

$$I(t) = -\int_{s_{i-1}}^{t} \beta(t_i - u) f(u, p_i)\,\mathrm{d}u, \quad s_{i-1} \le t < t_i. \tag{4}$$

From (2), the present value of the holding cost in the ith cycle, denoted by HC_i, can be written as

$$\mathrm{HC}_i = c_h \int_{t_i}^{s_i} \mathrm{e}^{-rt} \int_t^{s_i} \mathrm{e}^{\theta(u-t)} f(u, p_i)\,\mathrm{d}u\mathrm{d}t, \quad i = 1, 2, \ldots, n. \tag{5}$$

From (4), the present value of the shortage cost due to shortages during $[s_{i-1}, t_i)$ is

$$\begin{aligned} \mathrm{SC}_i &= c_s \int_{s_{i-1}}^{t_i} \mathrm{e}^{-rt} \int_{s_{i-1}}^{t} \beta(t_i - u) f(u, p_i)\,\mathrm{d}u\mathrm{d}t \\ &= \frac{c_s}{r} \int_{s_{i-1}}^{t_i} \left(\mathrm{e}^{-rt} - \mathrm{e}^{-rt_i}\right) \beta(t_i - t) f(t, p_i)\,\mathrm{d}t, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{6}$$

The present value of the cost of lost sales during $[s_{i-1}, t_i)$ is

$$\mathrm{LC}_i = c_l \int_{s_{i-1}}^{t_i} \mathrm{e}^{-rt} \left[1 - \beta\left(t_i - t\right)\right] f\left(t, p_i\right) \mathrm{d}t, \quad i = 1, 2, \ldots, n. \tag{7}$$

The order quantity at t_i in the ith replenishment cycle is

$$Q_i = \int_{s_{i-1}}^{t_i} \beta\left(t_i - t\right) f\left(t, p_i\right) \mathrm{d}t + \int_{t_i}^{s_i} \mathrm{e}^{\theta(t-t_i)} f\left(t, p_i\right) \mathrm{d}t, \quad i = 1, 2, \ldots, n. \tag{8}$$

The present value of the purchase cost during the ith replenishment cycle is

$$\begin{aligned} \mathrm{PC}_i &= c_f \mathrm{e}^{-rt_i} + c_v\left(t_i\right) \mathrm{e}^{-rt_i} Q_i \\ &= c_f \mathrm{e}^{-rt_i} + c_v\left(t_i\right) \mathrm{e}^{-rt_i} \int_{s_{i-1}}^{t_i} \beta\left(t_i - t\right) f\left(t, p_i\right) \mathrm{d}t + c_v\left(t_i\right) \mathrm{e}^{-rt_i} \int_{t_i}^{s_i} \mathrm{e}^{\theta(t-t_i)} f\left(t, p_i\right) \mathrm{d}t, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{9}$$

and the present value of the sales revenue in the ith replenishment cycle is

$$\mathrm{SR}_i = p_i \mathrm{e}^{-rt_i} \int_{s_{i-1}}^{t_i} \beta\left(t_i - t\right) f\left(t, p_i\right) \mathrm{d}t + p_i \int_{t_i}^{s_i} \mathrm{e}^{-rt} f\left(t, p_i\right) \mathrm{d}t, \quad i = 1, 2, \ldots, n. \tag{10}$$

If n replenishment orders are placed in $[0, H]$, then the discounted total profit of the inventory system during the planning horizon H is

$$\begin{aligned} \mathrm{TP}(\mathbf{p}, \mathbf{t}, \mathbf{s}|n) &= \sum_{i=1}^{n} \left(\mathrm{SR}_i - \mathrm{PC}_i - \mathrm{HC}_i - \mathrm{SC}_i - \mathrm{LC}_i\right) \\ &= \sum_{i=1}^{n} \left(p_i - c_v\left(t_i\right)\right) A\left(p_i\right) \left[\mathrm{e}^{-rt_i} \int_{s_{i-1}}^{t_i} \beta\left(t_i - t\right) g(t) \mathrm{d}t + \int_{t_i}^{s_i} \mathrm{e}^{-rt} g(t) \mathrm{d}t\right] - \sum_{i=1}^{n} c_f \mathrm{e}^{-rt_i} \\ &\quad - c_v(t_i) \sum_{i=1}^{n} A\left(p_i\right) \left[\mathrm{e}^{-rt_i} \int_{t_i}^{s_i} \left[\mathrm{e}^{\theta(t-t_i)} - \mathrm{e}^{-r(t-t_i)}\right] g(t) \mathrm{d}t\right] \\ &\quad - c_h \sum_{i=1}^{n} A\left(p_i\right) \int_{t_i}^{s_i} \mathrm{e}^{-rt} \int_{t}^{s_i} \mathrm{e}^{\theta(u-t)} g(u) \mathrm{d}u \mathrm{d}t \\ &\quad - \frac{c_s}{r} \sum_{i=1}^{n} A\left(p_i\right) \int_{s_{i-1}}^{t_i} \left(\mathrm{e}^{-rt} - \mathrm{e}^{-rt_i}\right) \beta\left(t_i - t\right) g(t) \mathrm{d}t \\ &\quad - c_l \sum_{i=1}^{n} A\left(p_i\right) \int_{s_{i-1}}^{t_i} \mathrm{e}^{-rt} \left[1 - \beta\left(t_i - t\right)\right] g(t) \mathrm{d}t, \end{aligned} \tag{11}$$

where $\mathbf{p} = \{p_1, p_2, \ldots, p_n\}$, $\mathbf{s} = \{s_1, s_2, \ldots, s_{n-1}\}$ and $\mathbf{t} = \{t_1, t_2, \ldots, t_n\}$. Now, the problem is to determine $\mathbf{p}$, $\mathbf{t}$ and $\mathbf{s}$ so that $\mathrm{TP}(\mathbf{p}, \mathbf{t}, \mathbf{s}|n)$ is maximized. Hence, it is a $3n - 1$ decision-making problem for a retailer and the problem can be written as

$$\begin{aligned} &\text{Maximize} && \mathrm{TP}(\mathbf{p}, \mathbf{t}, \mathbf{s}|n) \\ &\text{subject to} && cv(t_i) < p_i < p_u, \quad i = 1, 2, \ldots, n \\ &&& s_{i-1} < t_i < s_i, \quad i = 1, 2, \ldots, n \\ &&& s_0 = 0, \qquad s_n = H. \end{aligned}$$

The formulated optimization model is a nonlinear programming with nonnegative constraints in its objective. Since it is difficult to solve analytically, we adopt an evolutionary computation algorithm to solve the problem. Evolutionary computation is a globally optimization technique where the aim is to improve the ability of individual to survive. In this paper, an algorithm based on particle swarm optimization (PSO) is proposed to find the optimal pricing and replenishment schedule.

4. Solution procedure

The particle swarm optimization (PSO) is an algorithm for finding optimal regions of complex search spaces through the interaction of individuals in a population of particles. It was proposed by Eberhart and Kennedy [23,24] and has been widely used in finding solutions for optimization problems. The PSO algorithm is inspired by social behavior of bird flocking or fish schooling. In PSO, the potential solutions, called particles, fly through the problem space by following the current optimum particles. Here we will give a short description of PSO proposed by Eberhart and Kennedy [23,24].

4.1. The background of particle swarm optimization

As stated before, PSO simulates the behavior of bird flocking. Assume that our search space is d-dimensional, PSO is initialized with a group of random particles (solutions) and then searches for optima by updating generations. During every iteration, each particle is updated by following the two best values. The first one is the best searching experience of the individual so far and it is called pbest. The other one is the best result obtained so far by any particle in the population and it is called gbest. When pbest and gbest are obtained, a particle updates its velocity and position based on (12) and (13):

$$v_{k+1}^{j} = wv_{k}^{j} + c_1 \times \text{rand}() \times (\text{pbest}_k^i - x_k^i) + c_2 \times \text{rand}() \times (\text{gbest}_k - x_k^i) \tag{12}$$

and

$$x_{k+1}^{i} = x_k^i + v_{k+1}^i, \tag{13}$$

where

v_k^i: the velocity of i-th particle at the k-th iteration
x_k^i: the current position of i-th particle at the k-th iteration
pbest_k^i: the best searching experience so far by particle i at the k-th iteration
gbest_k: the best result obtained at the k-th iteration
c_1, c_2: the constant weight factor
rand(): random number between 0 and 1
w: the inertia weight.

Lastly, the algorithm will check the results every iteration until the best solution is found or termination conditions are satisfied.

The parameters c_1 and c_2 in (12) are scalar constants that weight influence of particles' own experience and the social knowledge. The parameter w in (12) maintains the previous flight direction of the particle to the personal best particle or global particle. Usually c_1 and c_2 are in the range from 1.5 to 2.5. Furthermore, suitable correction of inertia w in (12) provides a balance between global and local explorations, thereby reducing the number of iterations when finding a sufficiently optimal solution. The following linearly decreasing inertia weight function is usually utilized:

$$w = w_{\max} - \frac{w_{\max} - w_{\min}}{\text{iter}_{\max}} \times \text{iter}, \tag{14}$$

where $w_{\max}$ is the initial weight, $w_{\min}$ is the final weight, $\text{iter}_{\max}$ is the maximum iteration number, and iter is the current iteration number. Usually $w_{\max}$ and $w_{\min}$ are set to 0.9 and 0.4, respectively.

Furthermore, in the PSO algorithm, velocities are clamped at each iteration to lie within $[-V_{\max}, V_{\max}]$ on each dimension, which is a parameter specified by the user. If the sum of accelerations causes the velocity on that dimension to exceed $V_{\max}$, then this velocity is limited to $V_{\max}$. This helps particles comb the search space rather than potentially taking huge iterative steps that might cause some information to be missed. The search procedure of the particle swarm optimization is summarized as follows:

Step 1. Initialize a population of particles with random positions and velocities on d-dimensions in the search space.
Step 2. Evaluate the fitness of all particles.
Step 3. Keep track of the locations where each individual had its highest fitness so far.
Step 4. Keep track of the position with the global best fitness.
Step 5. Update the velocity of each particle, according to (12) and (14).
Step 6. Update the position of each particle, according to (13).
Step 7. Terminate if the criteria are satisfied, otherwise go to Step 2.

4.2. Solving the pricing and replenishment scheduling problem

For any given feasible replenishment schedule, $0 < t_1 < s_1 < \cdots < s_{n-1} < t_n < H$, to acquire optimal selling prices that maximize $\text{TP}(\mathbf{p}|n, \mathbf{t}, \mathbf{s})$, the values of p_i should be selected to satisfy

$$\begin{aligned}\frac{\partial \text{TP}(\mathbf{p}|n, \mathbf{t}, \mathbf{s})}{\partial p_i} &= \left[A(p_i) + (p_i - c_v(t_i))A'(p_i)\right]\left[\mathrm{e}^{-rt_i}\int_{s_{i-1}}^{t_i} \beta(t_i - t)g(t)\mathrm{d}t + \int_{t_i}^{s_i} \mathrm{e}^{-rt} g(t)\mathrm{d}t\right] \\ &\quad - c_v(t_i)A'(p_i)\left[\mathrm{e}^{-rt_i}\int_{t_i}^{s_i}\left[\mathrm{e}^{\theta(t-t_i)} - \mathrm{e}^{-r(t-t_i)}\right]g(t)\mathrm{d}t\right] - c_h A'(p_i)\int_{t_i}^{s_i}\mathrm{e}^{-rt}\int_t^{s_i}\mathrm{e}^{\theta(u-t)}g(u)\mathrm{d}u\mathrm{d}t \\ &\quad - \frac{c_s}{r}A'(p_i)\int_{s_{i-1}}^{t_i}\left(\mathrm{e}^{-rt} - \mathrm{e}^{-rt_i}\right)\beta(t_i - t)g(t)\mathrm{d}t - c_l A'(p_i)\int_{s_{i-1}}^{t_i}\mathrm{e}^{-rt}\left[1 - \beta(t_i - t)\right]g(t)\mathrm{d}t \\ &= 0, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{15}$$

Here, we can find that the necessary conditions are independent. If the gross profit, $(p_i - c_v(t_i))A(p_i)$, is a strictly concave function of p_i, we then have

$$\begin{aligned}\frac{\partial^2 \mathrm{TP}(\mathbf{p}|n,\mathbf{t},\mathbf{s})}{\partial p_i^2} &= \left[2A'(p_i) + (p_i - c_v(t_i))A''(p_i)\right]\left[\mathrm{e}^{-rt_i}\int_{s_{i-1}}^{t_i}\beta(t_i - t)g(t)\mathrm{d}t + \int_{t_i}^{s_i}\mathrm{e}^{-rt}g(t)\mathrm{d}t\right] \\ &\quad - c_v(t_i)A''(p_i)\left[\mathrm{e}^{-rt_i}\int_{t_i}^{s_i}\left[\mathrm{e}^{\theta(t-t_i)} - \mathrm{e}^{-r(t-t_i)}\right]g(t)\mathrm{d}t\right] \\ &\quad - c_h A''(p_i)\int_{t_i}^{s_i}\mathrm{e}^{-rt}\int_t^{s_i}\mathrm{e}^{\theta(u-t)}g(u)\mathrm{d}u\mathrm{d}t - \frac{c_s}{r}A''(p_i)\int_{s_{i-1}}^{t_i}\left(\mathrm{e}^{-rt} - \mathrm{e}^{-rt_i}\right)\beta(t_i - t)g(t)\mathrm{d}t \\ &\quad - c_l A''(p_i)\int_{s_{i-1}}^{t_i}\mathrm{e}^{-rt}\left[1 - \beta(t_i - t)\right]g(t)\mathrm{d}t \\ &< 0 \end{aligned} \tag{16}$$

and

$$\frac{\partial^2 \mathrm{TP}(\mathbf{p}|n,\mathbf{t},\mathbf{s})}{\partial p_i \partial p_j} = 0, \quad i \neq j,\ i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n. \tag{17}$$

As the discussion above, the Hessian matrix at the stationary point $\left(p_1^{\mathrm{opt}}, p_2^{\mathrm{opt}}, \ldots, p_n^{\mathrm{opt}}\right)$, denoted by $\mathbf{p}^{\mathrm{opt}}$, is given by

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial^2 \mathrm{TP}(\mathbf{p}|n,\mathbf{t},\mathbf{s})}{\partial p_1^2} & 0 & 0 & 0 \\ 0 & \dfrac{\partial^2 \mathrm{TP}(\mathbf{p}|n,\mathbf{t},\mathbf{s})}{\partial p_2^2} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dfrac{\partial^2 \mathrm{TP}(\mathbf{p}|n,\mathbf{t},\mathbf{s})}{\partial p_n^2} \end{bmatrix}.$$

We can see that the diagonal elements of $\mathbf{H}$ are all negative and off-diagonal elements are all zero, and thus, the Hessian matrix $\mathbf{H}$ at point $\mathbf{p}^{\mathrm{opt}}$ is negative definite and $\mathbf{p}^{\mathrm{opt}}$ represents a global maximum point.

From the analysis carried out so far, it is easy to see that $p_i^{\mathrm{opt}} = p_i^*$ can be written as a function of $\mathbf{t}$ and $\mathbf{s}$, and this result reduces the $3n - 1$ dimensional problem of finding the optimal pricing and schedule to a $2n - 1$ dimensional problem as follows:

$$\begin{aligned} &\text{Maximize} && \mathrm{TP}(\mathbf{t},\mathbf{s}|n) \\ &\text{subject to} && cv(t_i) < p_i^{\mathrm{opt}} < p_u, \quad i = 1, 2, \ldots, n \\ &&& s_{i-1} < t_i < s_i, \quad i = 1, 2, \ldots, n \\ &&& s_0 = 0, \qquad s_n = H. \end{aligned}$$

The above constrained maximization problem can be transformed into an unconstrained one by adding a penalty term, which can transform the constrained objective function to a pseudo objective function as follows:

$$\begin{aligned}\phi(\mathbf{t},\mathbf{s}|n) &= \mathrm{TP}(\mathbf{t},\mathbf{s}|n) + \mu\sum_{i=1}^{n}\Big\{\left\{\max\left[0, p_i^{\mathrm{opt}} - p_u\right]\right\}^2 + \left\{\max\left[0, cv(t_i) - p_i^{\mathrm{opt}}\right]\right\}^2 \\ &\quad + \{\max[0, s_{i-1} - t_i]\}^2 + \{\max[0, t_i - s_i]\}^2\Big\}, \end{aligned} \tag{18}$$

where μ is a large positive number, known as the penalty number. Eq. (18) is then used to evaluate the fitness of individuals in a population. Thus, for any given integer of n, the problem becomes

$$\begin{aligned} &\text{Maximize} && \phi(\mathbf{t},\mathbf{s}|n) \\ &\text{subject to} && s_0 = 0, \qquad s_n = H \end{aligned}$$

and the solution procedure for finding optimal pricing and replenishment schedule is provided as follows.

Step 1. Let dimension $d = 2n - 1$, population size $I = 10d$, $V_{\max} = H/(1.5d)$, $c_1 = c_2 = 2.05$, $\mu = 10^9$, $\mathrm{iter}_{\max} = 1000$ and $k = 0$.

Step 2. x_0^i: Randomly generate and sort d points in the range 0 to H, $i = 1, 2, \ldots, I$.

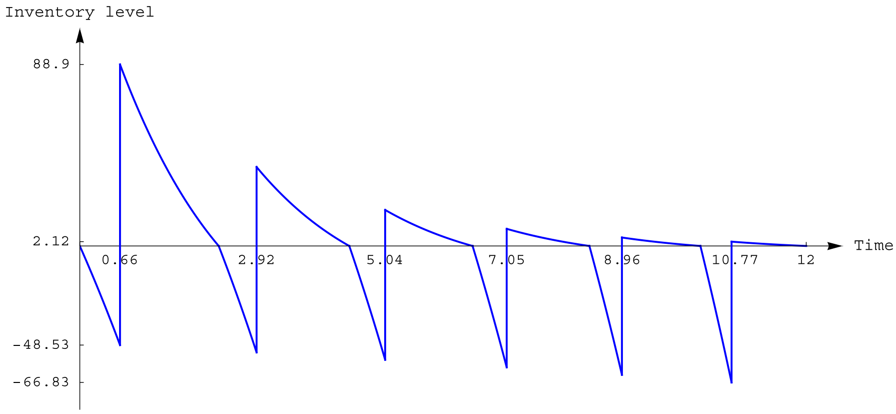


Fig. 2. Graphical representation of inventory system for Example 1.

Step 3. v_0^i: Randomly generate d points in the range $-V_{\max}$ and $V_{\max}$, $i = 1, 2, \ldots, I$.
Step 4. Use (15) to compute p_i^{opt} and evaluate the fitness of all particles from (18).
Step 5. Compute pbest_k^i, $i = 1, 2, \ldots, I$.
Step 6. Compute gbest_k for this iteration.
Step 7. Update v_k^i, $i = 1, 2, \ldots, I$, according to (12) and (14).
Step 8. Update x_{k+1}^i, $i = 1, 2, \ldots, I$, according to (13).
Step 9. Terminate if the standard deviation of $\phi(\mathbf{t}, \mathbf{s}|n) < 10^{-5}$ or $k = \text{iter}_{\max}$, otherwise $k = k + 1$ and go to *Step* 4.

To avoid using a brute force enumeration for finding n^*, we further simplify the search process by providing an intuitively good starting value for n^*. It is clear from (15) that $\partial \text{TP}(\mathbf{p}|n, \mathbf{t}, \mathbf{s})/\partial p_i = 0$ has a solution if $A(p_i) + (p_i - c_v(t_i))A'(p_i) < 0$. As a result, the solution of $A(p) + (p - c_v(t))A'(p) = 0$, say p^l, is the lower bound for the optimal selling price. Using the Eq. (16) in [18], we can obtain an estimate of the number of replenishments as

$$n = \text{round integer of} \sqrt{\frac{(c_h + \bar{c}_v\theta)\left[\beta(1)c_s + [1-\beta(1)](p^l - \bar{c}_v + c_l)\right] Q\left(H|p^l\right) H}{2c_f\left[c_h + \bar{c}_v\theta + \beta(1)c_s + [1-\beta(1)](p^l - \bar{c}_v + c_l)\right]}}, \tag{19}$$

where $\bar{c}_v$ is the average unit cost, p^l is the solution of $A(p) + (p - \bar{c}_v)A'(p) = 0$ and $Q(H|p^l) = \int_0^H A\left(p^l\right) g(t)\mathrm{d}t$. It is obvious that searching for the optimal number of replenishments by starting with n in (19) instead of $n = 1$ will reduce the computational complexity significantly.

Combining the above arguments, we propose the following algorithm to solve the pricing and replenishment scheduling problem.

4.3. Algorithm 2

Step 1. Choose two initial trial values of n^*, say n as in (19) and $n - 1$. Use Algorithm 1 to obtain $\mathbf{p}^*$, $\mathbf{t}^*$ and $\mathbf{s}^*$, and compute the corresponding $\text{TP}(n)$ and $\text{TP}(n-1)$, respectively.
Step 2. If $\text{TP}(n) \le \text{TP}(n-1)$, then compute $\text{TP}(n-2), \text{TP}(n-3), \ldots,$ until we find $\text{TP}(k) > \text{TP}(k-1)$. Set $n^* = k$ and stop.
Step 3. If $\text{TP}(n) > \text{TP}(n-1)$, then compute $\text{TP}(n+1), \text{TP}(n+2), \ldots,$ until we find $\text{TP}(k) > \text{TP}(k+1)$. Set $n^* = k$ and stop.

5. Computational results

5.1. Numerical examples

To illustrate the results, let us apply the proposed algorithms to solve the following numerical examples. Algorithms 1 and 2 are implemented on a personal computer with Intel Core 2 Duo under Mac OS X 10.5.6 operating system using Mathematica version 7.

Example 1. We first redo the same example of Chen and Chen [21] while considering the time increasing demand. $f(t, p) = (300 - 120p)\,\mathrm{e}^{0.06t}$, $c_f = 40$, $c_h = 0.02$, $c_s = 0.5$, $\theta = 0.2$, $H = 12$, $r = 0.02$. Besides, we assume that the time-dependent backlogging rate is $c_v(t) = \mathrm{e}^{0.01t}$, $\beta(x) = \mathrm{e}^{-0.2x}$ and take $c_l = 0.6$. By applying (19), we obtain the estimated number of replenishments $n = 6$. Then, applying Algorithms 1 and 2, we get $\text{TP}(5) = 474.1$, $\text{TP}(6) = 487.4$ and $\text{TP}(7) = 487.2$. Therefore, the optimal number of replenishments is 6, and the optimal pricing and time schedule are shown in Table 1. The behavior of inventory system over the planning horizon and the convergence result of PSO algorithms for optimal solution are depicted in Figs. 2 and 6(a), respectively.

Table 1
Optimal pricing and time schedule for Example 1.

i	p_i	$cv(t_i)$	t_i	s_i	Q_i	LS_i[a]	LI_i[b]
1	1.8630	1.0067	0.6643	2.2946	137.4	0.6643	1.6303
2	1.8688	1.0296	2.9202	4.4519	90.8	0.6256	1.5317
3	1.8750	1.0517	5.0443	6.4886	73.4	0.5924	1.4443
4	1.8812	1.0731	7.0520	8.4184	67.8	0.5634	1.3664
5	1.8876	1.0937	8.9559	10.2523	67.2	0.5375	1.2964
6	1.8940	1.1137	10.7670	12.0000	68.9	0.5147	1.2330

[a] $LS_i = t_i - s_{i-1}$.
[b] $LI_i = s_i - t_i$.

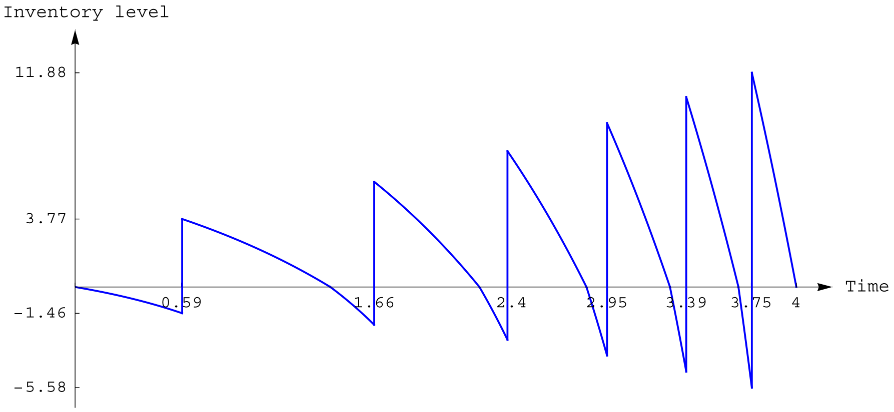


Fig. 3. Graphical representation of inventory system for Example 2.

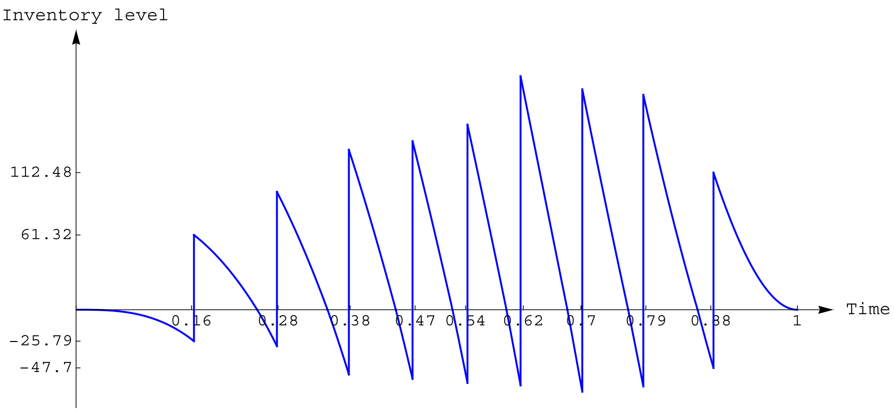


Fig. 4. Graphical representation of inventory system for Example 3.

Example 2. In this example, we redo an inventory situation proposed by Teng and Yang [18] while considering $f(t,p) = (5 - 0.005p)e^{0.98t}$, $\beta(x) = e^{-0.2x}$, $c_f = 250$, $c_h = 40$, $c_s = 50$, $c_l = 500$, $\theta = 0.08$, $c_v(t) = 200 + 20e^{-2t}$, $H = 4$ and $r = 0.02$. By applying (19), we obtain the estimated number of replenishments $n = 6$. Then, apply Algorithms 1 and 2 to get TP(5) = 35 195.7, TP(6) = 35 217.4 and TP(7) = 35 163.8. Therefore, the optimal number of replenishments is 6, and the optimal pricing and time schedule is shown in Table 3. The behavior of inventory system over the planning horizon and the convergence result of PSO algorithms for optimal solution are depicted in Figs. 3 and 6(b), respectively.

Example 3. In this example, we redo an inventory situation proposed by Chen et al. [12] while considering the variable purchase cost, deterioration and partial backlogging. $f(t,p) = (5000 - 150p)\, t^{3-1} (H-t)^{2-1} / \mathcal{B}(3,2)$, $c_f = 50$, $c_h = 5$, $c_s = 7$, $c_l = 6$, $c_v(t) = 10e^{-0.25t}$, $H = 1$, $r = 0.02$, $\theta = 0.08$ and $\beta(x) = 1/(1+x)$. Applying (19), we obtain the estimated number of replenishments $n = 8$. Then, applying the Algorithms 1 and 2, we get TP(7) = 21 654.2, TP(8) = 21 680.1, TP(9) = 21 691.9 and TP(10) = 21 689.8. Therefore, the optimal number of replenishments is 9, and the optimal pricing and time schedule is shown in Table 3. The behavior of the inventory system over the planning horizon and the convergence result of PSO algorithms for optimal solution are depicted in Figs. 4 and 6(c), respectively.

Table 2
Optimal pricing and time schedule for Example 2.

i	p_i	$cv(t_i)$	t_i	s_i	Q_i	LS_i[a]	LI_i[b]
1	619.023	206.100	0.5938	1.4155	5.2	0.5938	0.8218
2	609.932	200.725	1.6590	2.2426	7.9	0.2434	0.5836
3	606.923	200.165	2.3980	2.8360	10.5	0.1555	0.4379
4	605.339	200.055	2.9502	3.2992	12.9	0.1143	0.3490
5	604.344	200.023	3.3895	3.6789	15.2	0.0903	0.2894
6	603.657	200.011	3.7533	4.0000	17.5	0.0744	0.2467

[a] $\mathrm{LS}_i = t_i - s_{i-1}$.
[b] $\mathrm{LI}_i = s_i - t_i$.

Table 3
Optimal pricing and time schedule for Example 3.

i	p_i	$cv(t_i)$	t_i	s_i	Q_i	LS_i[a]	LI_i[b]
1	21.6551	9.5927	0.1663	0.2629	98.9	0.1663	0.0966
2	21.4171	9.2795	0.2991	0.3695	150.0	0.0362	0.0704
3	21.2898	9.0557	0.3968	0.4613	190.2	0.0273	0.0645
4	21.1804	8.8596	0.4843	0.5427	209.8	0.0231	0.0583
5	21.0879	8.6860	0.5635	0.6189	221.1	0.0208	0.0554
6	21.0057	8.5235	0.6390	0.6947	229.4	0.0201	0.0557
7	20.9270	8.3680	0.7127	0.7696	218.9	0.0180	0.0569
8	20.8562	8.2092	0.7893	0.8545	213.0	0.0197	0.0652
9	20.7954	8.0334	0.8759	1.0000	175.0	0.0214	0.1241

[a] $\mathrm{LS}_i = t_i - s_{i-1}$.
[b] $\mathrm{LI}_i = s_i - t_i$.

Table 4
Optimal pricing and time schedule for Example 4.

i	p_i	$cv(t_i)$	t_i	s_i	Q_i	LS_i[a]	LI_i[b]
1	431.821	198.943	0.1060	0.7100	10.4	0.1060	0.6040
2	418.718	191.963	0.8203	1.4476	9.2	0.1103	0.6274
3	406.133	184.969	1.5626	2.2252	8.1	0.1150	0.6625
4	393.983	177.851	2.3474	3.0606	6.9	0.1222	0.7132
5	383.163	170.486	3.1932	4.0000	5.8	0.1326	0.8068

[a] $\mathrm{LS}_i = t_i - s_{i-1}$.
[b] $\mathrm{LI}_i = s_i - t_i$.

Example 4. In this example, we consider an inventory situation with $f(t, p) = 30\,000p^{-2} \times (100 - 15t)$, $c_f = 250$, $c_h = 40$, $c_s = 80$, $c_l = 120$, $c_v(t) = 200e^{-0.05t}$, $H = 4$, $r = 0.02$, $\theta = 0.08$ and $\beta(x) = 1/(1 + x)$. Applying (19), we obtain the estimated number of replenishments $n = 5$. Then, applying the Algorithms 1 and 2, we get $\mathrm{TP}(4) = 8606.1$, $\mathrm{TP}(5) = 8629.3$ and $\mathrm{TP}(6) = 8572.8$. Therefore, the optimal number of replenishments is 5, and the optimal pricing and time schedule is shown in Table 4. The behavior of inventory system over the planning horizon and the convergence results of PSO algorithms for an optimal solution are depicted in Figs. 5 and 6(d), respectively.

The following inferences can be made from the results in Tables 1–4.

1. When the demand is increasing with time, the length of shortages period (LS_i) in the ith replenishment cycle is decreasing. Otherwise, the length of shortages period (LS_i) is increasing.
2. When the demand is increasing with time, the length of inventory period (LI_i) in the ith replenishment cycle is decreasing. Otherwise, the length of inventory period (LI_i) is increasing.
3. When the unit purchasing cost is increasing with time, the selling price p_i in the ith replenishment cycle is increasing. Otherwise, the selling price p_i is decreasing.

5.2. Sensitivity analyses

Using the settings in the examples above, we perform the sensitivity analysis to investigate the effects of several factors such as ordering cost (c_f), holding cost (c_h), shortages cost (c_s), cost of lost sale (c_l) and deteriorating rate (θ) on the optimal discounted total profit (TP^*) and the optimal number of replenishments (n^*). The numerical results obtained are shown in Table 5, and are portrayed in Fig. 7. The following inferences are made based on the results in Table 5 and Fig. 7.

Table 5
Sensitivity analysis on TP* and n^* for Examples 1–4.

	Parameters	% change				
		−50%	−25%	+0%	+25%	+50%
Example 1	c_f	623.6 9	549.5 7	487.4 6	434.0 6	384.9 5
	c_h	494.1 6	490.7 6	487.4 6	484.4 7	481.5 7
	c_s	515.4 6	499.6 6	487.4 6	479.0 7	472.2 7
	c_l	493.2 6	490.2 6	487.4 6	485.0 7	482.9 7
	θ	580.3 5	529.0 6	487.4 6	455.7 7	428.0 7
Example 2	c_f	9314.1 7	8935.5 6	8629.3 5	8363.6 4	8121.1 4
	c_h	8958.4 4	8775.4 4	8629.3 5	8496.4 5	8372.1 5
	c_s	8647.8 5	8638.1 5	8629.3 5	8621.2 5	8613.9 5
	c_l	8658.6 5	8642.8 5	8629.3 5	8617.5 5	8607.1 5
	θ	8740.6 4	8684.3 5	8629.3 5	8575.4 5	8522.6 5
Example 3	c_f	21 960.1 12	21 810.4 10	21 691.9 9	21 581.5 8	21 482.9 8
	c_h	21 816. 8	21 749.6 9	21 691.9 9	21 640.4 10	21 597.7 10
	c_s	21 715.7 9	21 701.6 9	21 691.9 9	21 681.6 10	21 673.3 10
	c_l	21 710.4 9	21 701.5 9	21 691.9 9	21 682.2 9	21 673.8 10
	θ	21 707.3 9	21 698.9 9	21 691.9 9	21 684. 9	21 675.2 10
Example 4	c_f	36 007.3 8	35 580.3 7	35 217.4 6	34 898.1 5	34 600.4 5
	c_h	35 581.7 5	35 380.0 5	35 217.4 6	35 075. 6	34 943.6 6
	c_s	35 255.4 6	35 235.5 6	35 217.4 6	35 200.1 6	35 184.4 6
	c_l	35 302.1 6	35 256.2 6	35 217.4 6	35 181.1 6	35 155.0 6
	θ	35 350.0 5	35 280.6 6	35 217.4 6	35 152.5 6	35 096.1 6

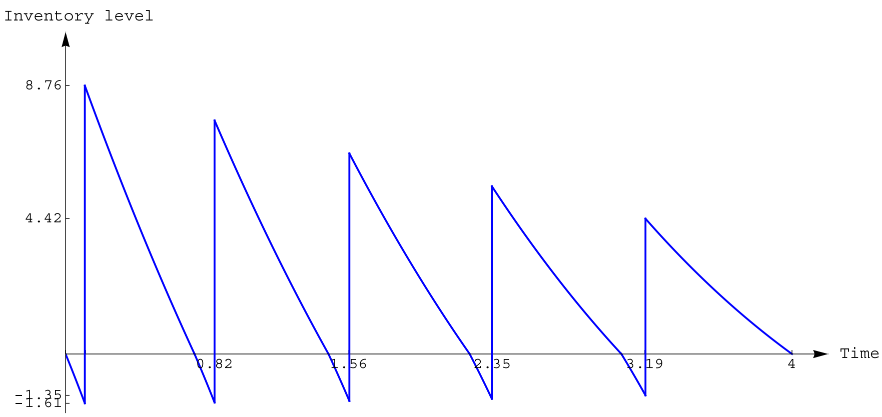
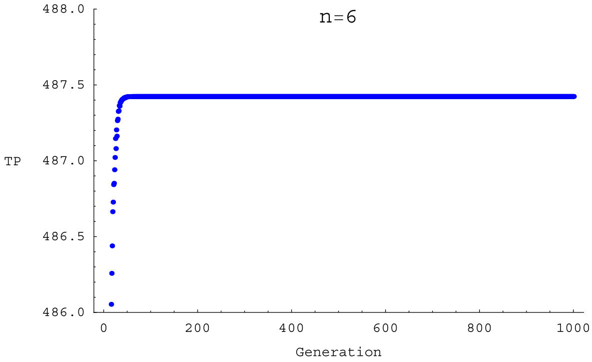
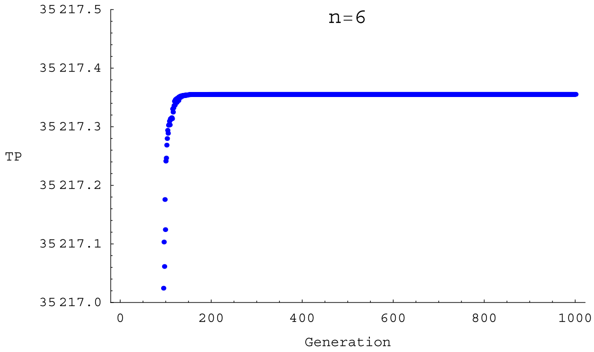


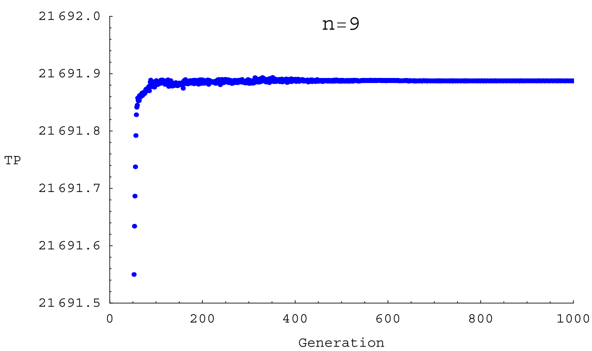
Fig. 5. Graphical representation of inventory system for Example 4.



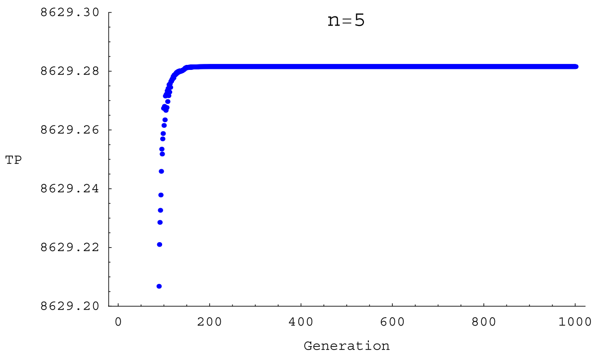
(a) Example 1.



(b) Example 2.



(c) Example 3.



(d) Example 4.

Fig. 6. Convergence results of PSO algorithm for TP.

1. The discounted total profit increases if c_f, c_h, c_s, c_l or θ decreases.
2. When the unit purchasing cost is increasing with time such as Example 1, the discounted total profit is more sensitive on the change in c_f or θ.
3. When the unit purchasing cost is decreasing with time such as Examples 2–4, the discounted total profit is not sensitive to the change in c_f, c_h, c_s, c_l or θ.
4. The number of replenishments increases as c_h, c_s, c_l or θ increases, while it decreases as c_f increases.

6. Concluding remarks

In this paper, an inventory control problem is developed with a deteriorating item, partial backordering of shortages, a time-varying item purchase cost, and a deterministic demand rate that depends on both time and price. The PSO algorithm is used to search for the optimal replenishment strategy, and the price within each replenishment cycle. In contrast to the classical fixed selling price policy under fixed unit purchasing cost, the pricing policy in this model is more flexible by changing the price upward or downward periodically. Consequently, the model is more suitable for managers to plan marketing strategies to address the challenges that their products are likely to face. Furthermore, from the convergence results, the PSO algorithm offers acceptable efficiency and accurate search capability.

The proposed model can be extended in several ways. For instance, we may consider the permissible delay in payments. Also, we could extend the deterministic demand function to price and stock-dependent demand patterns. Finally, we could generalize the model to allow for quantity discounts, finite capacity and others.

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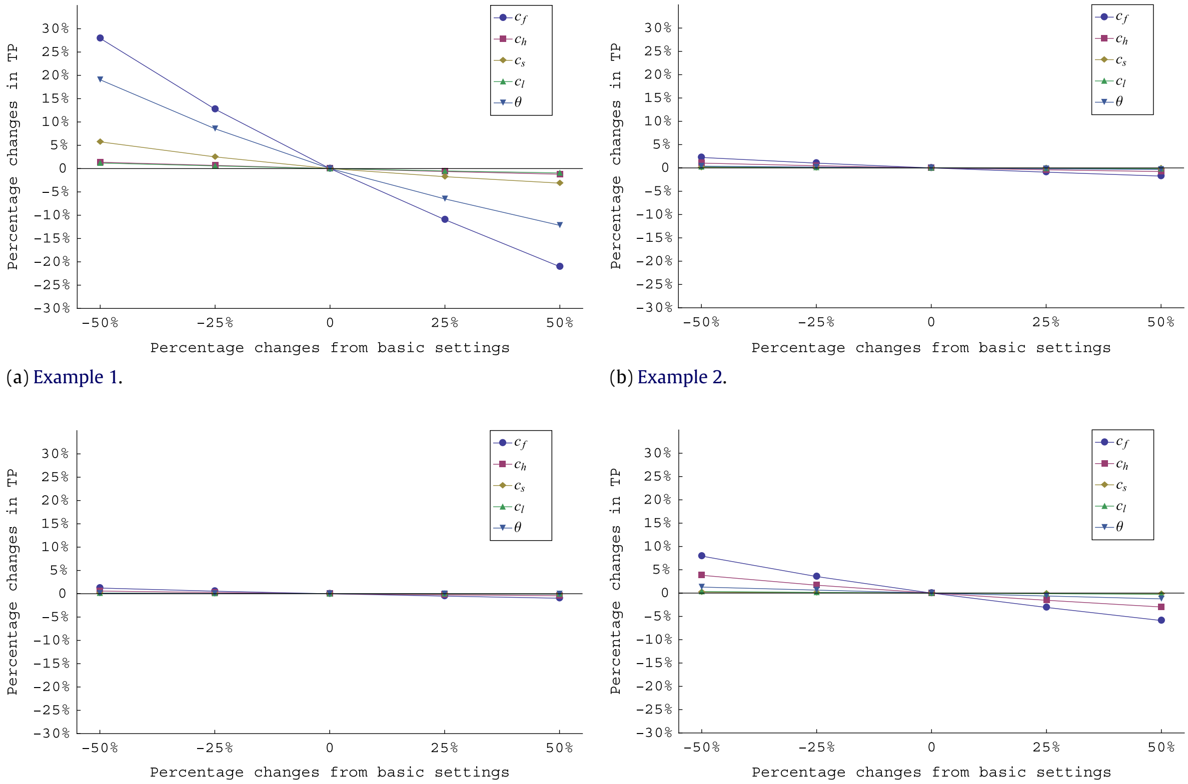

(a) Example 1.

(b) Example 2.

(c) Example 3.

(d) Example 4.

Fig. 7. Effects of c_f, c_h, c_s, c_l, and θ on the discounted total profit.

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